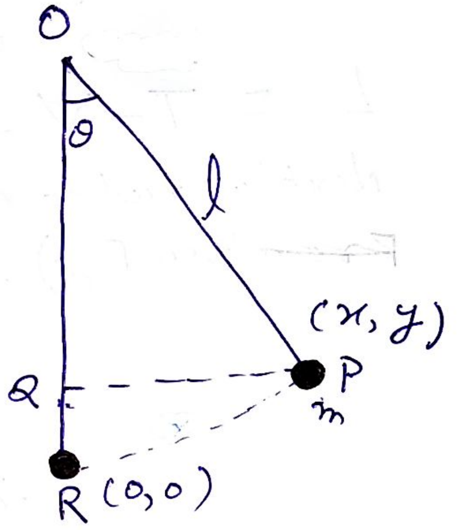
Examples on Lagrange's equations

(Q.1) Obtain the equation of motion of a simple pendulum by using Lagrangian method and hence deduce the formula for its time period for small amplitude oscillations.

Soln.



· In figure:

$\theta \rightarrow$ angular displacement of the simple pendulum from the equilibrium

$l \rightarrow$ Effective length of the pendulum

$m \rightarrow$ mass of the bob.

∴ Since this is a conservative system, Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{--- (i)}$$

$q_k \rightarrow$ generalized coordinates (Independent coordinates)

Let the point R is the origin (0,0) and P has coordinates (x, y) in cartesian coordinate system.

Since this system has a constraint on the motion of the particle; therefore, the generalized coordinates or the no. of independent coordinate will be $2-1=1$

In Cartesian coordinate system,

Kinetic energy of the system at point P is given by $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$ —— (2)

Potential energy $V = mg(RQ)$ {Let V at R is zero}

or $V = mgy$ —— (3)

$L = T - V$ —— (4)

Next, we need to obtain above expressions in terms of generalized coordinate θ.

(See Fig.). $x = QP = l\sin\theta$

$$y = RQ = l - QO = l - l\cos\theta.$$

$$\dot{x} = l\cos\theta\,\dot{\theta}$$

$$\dot{y} = -l\sin\theta\,\dot{\theta}$$

$$\dot{x}^2+\dot{y}^2 = l^2\dot{\theta}^2 - 2l^2\cos\theta\sin\theta\,\dot{\theta}^2 \simeq l^2\dot{\theta}^2 - 2l^2\theta\dot{\theta}^2$$

for small amplitude θ we take the above approximation ($\sin\theta\simeq\theta$, $\cos\theta\simeq 1-\frac{\theta^2}{2}\simeq 1$) and neglect the second term.

$$\dot{x}^2+\dot{y}^2 = l^2\dot{\theta}^2$$

$\therefore$ From (2) $T = \frac{1}{2}ml^2\dot{\theta}^2$

From (3). $V = mg(l - l\cos\theta) = mgl(1-\cos\theta)$

Therefore, $L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$ —— (5)

Eq (1) in terms of generalized coordinate. θ is written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{—— (6)}$$

From (5) $\quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

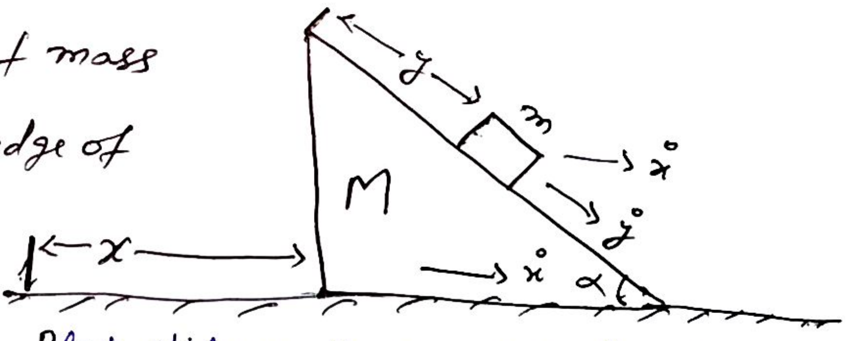
or $\quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$

From small amplitude oscillations, $\sin\theta \simeq \theta$

$$\therefore \quad \boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

Time period $\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$

H.W. Consider a block of mass m sliding on a smooth wedge of mass M and angle α which itself slides on a smooth horizontal floor, see Fig; The whole ~~moth~~ motion is planar. Find Lagrange's equations for this system and deduce (i) the acceleration of the wedge, and (ii) the acceleration of the block relative to the wedge.



Block slides on the smooth surface of the wedge which slides on a smooth ~~horizontal~~ horizontal floor.

Hint: See Fig. K.E., $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\alpha)$

P.E., $V = -mgy\sin\alpha$